

THE HAUSMANN-WEINBERGER 4-MANIFOLD INVARIANT OF ABELIAN GROUPS

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ABSTRACT. The Hausmann-Weinberger invariant of a group G is the minimal Euler characteristic of a closed orientable 4-manifold M with fundamental group G . We compute this invariant for finitely generated free abelian groups and estimate the invariant for all finitely generated abelian groups.

1. INTRODUCTION

For any finitely presented group G there exists a closed oriented 4-manifold M with $\pi_1(M) = G$. Hausmann and Weinberger [HW] defined the integer valued invariant $q(G)$ to be the least Euler characteristic among all such M . The explicit construction of a 4-manifold with $\pi_1(M) = G$, based on a presentation of G , yields an upper bound on $q(G)$. As pointed out in [HW], the isomorphism $H_1(M) \rightarrow H_1(G)$, the surjection $H_2(M) \rightarrow H_2(G)$, and Poincaré duality yield a lower bound. Together these bounds are

$$2 - 2\beta_1(G) + \beta_2(G) \leq q(G) \leq 2 - 2\text{def}(G), \quad (1.1)$$

where $\text{def}(G)$ is the deficiency of G , the maximum possible difference $g - r$ where the g is the number of generators and r the number of relations in a presentation of G , and $\beta_i(G)$ denotes the i th Betti number of G (with some fixed coefficients).

Since [HW], advances have been made in the study of this invariant, most notably through the methods of l^2 -homology. For instance, in [E1, E2] Eckmann proves that for infinite amenable groups G , $q(G) \geq 0$. Lück [Lü] extended this to all group G with $b_1(G) = 0$, where b_1 denotes the first l^2 -Betti number. Other work includes [JK] and especially the paper by Kotschick [K] in which Problem 5.2 asks for the explicit value of $q(\mathbf{Z}^n)$. The general problem of computing $q(G)$ appears as Problem 4.59 in Kirby's problem list, [Ki].

Despite these past efforts, the Hausmann-Weinberger invariant remains uncomputed for some of the most elementary groups. In [HW] it is observed that $q(\mathbf{Z}^n)$ is given by 2, 0, 0, 2, and 0, for $n = 0, 1, 2, 3$, and 4, respectively. (For the case of $n = 3$, [HW] refers to an unpublished argument of Kreck. Proofs appear in [E1, K].) If Γ_g denotes the fundamental group of a surface of genus g , [K] computes $q(\Gamma_g)$ and $q(\Gamma_{g_1} \times \Gamma_{g_2})$. If a closed 4-manifold X is aspherical then $\chi(X) = q(\pi_1(X))$. Beyond this, few explicit values of $q(G)$ have been calculated. Our main theorem is the following:

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Theorem 1. *With the exceptions of $q(\mathbf{Z}^3) = 2$ and $q(\mathbf{Z}^5) = 6$, $q(\mathbf{Z}^n)$ is given by:*

$$q(\mathbf{Z}^n) = \begin{cases} (n-1)(n-4)/2, & \text{if } n \equiv 0 \text{ or } n \equiv 1 \pmod{4}; \\ (n-2)(n-3)/2, & \text{if } n \equiv 2 \text{ or } n \equiv 3 \pmod{4}. \end{cases}$$

For contrast, the bounds (1.1) give only that

$$(n-1)(n-4)/2 \leq q(\mathbf{Z}^n) \leq (n-1)(n-2).$$

It is straightforward to check that an alternative way to state Theorem 1 is that for $n \neq 3, 5$, the lower bound of (1.1) is attained when the binomial coefficient $C(n, 2)$ is even, and if $C(n, 2)$ is odd the value is one more than the lower bound of (1.1).

The calculations of $q(\mathbf{Z}^n)$ can be used to estimate (and in some cases calculate) $q(G)$ for other groups. We examine the problem for finitely generated abelian groups and prove the following. In the statement, ϵ_n equals zero or one according to whether $C(n, 2)$ is even or odd, respectively.

Theorem 2. *Let $G = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k \oplus \mathbf{Z}^n$ with $d_i | d_{i+1}$. Suppose that $k \geq 1$ and $k + n \neq 3, 4, 5$ or 6. Then*

$$0 \leq q(G) - (1 - n + C(n + k - 1, 2)) \leq \min\{|n - 1| + \epsilon_{n+k-1}, k + \epsilon_{n+k}\}.$$

Moreover, $q(\mathbf{Z}/d) = 2$, $q(\mathbf{Z}/d_1 \oplus \mathbf{Z}/d_2) = 2$, and if $C(k, 2)$ is even and $k \neq 5$, then

$$q(\mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k \oplus \mathbf{Z}) = C(k, 2).$$

In closing this introduction we mention results concerning the evaluation of $q(P)$ where P is a perfect group. Here the bounds given by (1.1) are

$$2 + \beta_2(P) \leq q(P) \leq 2 - 2\text{def}(G).$$

In [HW] perfect groups P are constructed with $\beta_2(P) = 0$ but $q(P) > 2$. Hillman [H] constructed perfect groups of deficiency -1 with $q(P) = 2$ and the second author [L] extended this to find perfect P with arbitrarily large negative deficiency and $q(P) = 2$.

2. NOTATION AND BASIC RESULTS

A slightly different invariant, $h(G)$, can be defined to be the minimum value of $\beta_2(M)$ among all oriented closed 4-manifolds M with $\pi_1(M) = G$. We abbreviate $h(\mathbf{Z}^n) = h(n)$. Clearly $q(G) = 2 - 2\beta_1(G) + h(G)$, so the invariants are basically equivalent. It is more convenient here to work in terms of h . The bounds (1.1) on $q(\mathbf{Z}^n)$ translate to the bounds

$$C(n, 2) \leq h(n) \leq 2C(n, 2).$$

We introduce the following auxiliary function:

$$\epsilon_n = \begin{cases} 0, & \text{if } C(n, 2) \text{ is even;} \\ 1, & \text{if } C(n, 2) \text{ is odd.} \end{cases}$$

Since $C(n, 2)$ is even if and only if $n \equiv 0$ or $1 \pmod{4}$, Theorem 1 can be restated as follows.

Theorem 1. *With the exceptions of $h(3) = 6$ and $h(5) = 14$, $h(n) = C(n, 2) + \epsilon_n$ for all n .*

Basic examples of 4-manifolds will be built from products of surfaces. For n even we will denote by F_n the closed orientable surface of genus $n/2$.

3. BOUNDS ON $h(n)$

Theorem 3. *If $h(n) = C(n, 2)$, then $C(n, 2)$ must be even. Thus $h(n) \geq C(n, 2) + \epsilon_n$.*

Proof. If $\phi: \pi_1(M) \rightarrow \mathbf{Z}^n$, we have $\phi_*: H_*(M) \rightarrow H_*(\mathbf{Z}^n)$. Dually there is the map of cohomology rings $\phi^*: H^*(\mathbf{Z}^n) \rightarrow H^*(M)$. Notice that $H^*(\mathbf{Z}^n)$ is an exterior algebra on the generators $e_1, \dots, e_n \in H^1(\mathbf{Z}^n)$.

Suppose $\phi: \pi_1(M) \rightarrow \mathbf{Z}^n$ is an isomorphism and $\beta_2(M) = C(n, 2)$. Then the map $\phi_2: H_2(M) \rightarrow H_2(\mathbf{Z}^n)$ is a surjection from $\mathbf{Z}^{C(n, 2)}$ to $\mathbf{Z}^{C(n, 2)}$, and hence is an isomorphism. It follows that $\phi^2: H^2(\mathbf{Z}^n) \rightarrow H^2(M)$ is also an isomorphism.

Since $(e_i e_j)^2 = 0$, $H^2(M)$ has a basis for which all squares are zero. It follows that the intersection form of M is even. But even unimodular forms are of even rank. \square

Theorem 4. *$h(3) \geq 6$ and $h(5) \geq 14$.*

Proof. In general, if $\phi: \pi_1(M) \rightarrow \mathbf{Z}^n$ is an isomorphism, then $\phi^2: H^2(\mathbf{Z}^n) \rightarrow H^2(M)$ is injective.

In the case that $n = 3$, all products of two elements in $H^2(\mathbf{Z}^3)$ are 0 (since $H^4(\mathbf{Z}^3) = 0$) so the intersection form on $H^2(M)$ vanishes on a rank 3 submodule, implying that this (nonsingular) form must have rank at least 6.

In the case that $n = 5$, we have the map $H^4(\mathbf{Z}^5) \rightarrow H^4(M) \cong \mathbf{Z}$. Any such map is given by multiplying with an element $D \in H^1(\mathbf{Z}^5)$. After a change of basis, D can be taken to be a multiple of a generator, say e_1 . From this it follows that the intersection form of $H^2(M)$ vanishes on the 7-dimensional submodule generated by the images of the set of elements in $H^2(\mathbf{Z}^5)$, $\{e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}\}$ (where $e_{ij} = e_i e_j$). To see this, observe that the only possible nontrivial products of two of these are $\pm e_{1234}, \pm e_{1235}$ and $\pm e_{1245}$, each of which is killed upon multiplying by e_1 . Since the nonsingular intersection form on $H^2(M)$ has a 7-dimensional isotropic subspace, it must be of rank at least 14. \square

4. ALGEBRAIC AND GEOMETRIC 4-REDUCTIONS.

The following algebraic construction will be used repeatedly in constructing our desired 4-manifolds.

Definition 5. A 4-reduction of a group G by a 4-tuple of elements $[w_1, w_2, w_3, w_4]$, $w_i \in G$, is the quotient of G by normal subgroup generated by the 6 commutators, $[w_i, w_j], i < j$. This quotient is denoted $G/[w_1, w_2, w_3, w_4]$.

More generally, we say a group G can be 4-reduced to the group H using the 4-tuples $\{[w_{1k}, w_{2k}, w_{3k}, w_{4k}]\}_{k=1}^\ell$ if H is isomorphic to the quotient of G by the normal subgroup generated by the 6ℓ commutators $[w_{ik}, w_{jk}], i < j, k = 1, \dots, \ell$.

The geometric motivation for this comes from the following theorem.

Theorem 6. *If X is a 4-manifold and $\{w_1, w_2, w_3, w_4\} \subset \pi_1(X)$, then there is a 4-manifold X' with $\pi_1(X') = \pi_1(X)/[w_1, w_2, w_3, w_4]$ and $\beta_2(X') = \beta_2(X) + 6$.*

Before proving this we make the following simple observation.

Lemma 7. *If a 4-manifold X' is constructed from a compact 4-manifold X via surgery along a curve α , then $\beta_2(X') = \beta_2(X)$ if α is of infinite order in $H_1(X)$ and $\beta_2(X') = \beta_2(X) + 2$ otherwise.*

Proof. Since X' is formed by removing $S^1 \times B^3$ and replacing it with $B^2 \times S^2$, $\chi(X') = \chi(X) + 2$. If α is of infinite order, $\beta_1(X') = \beta_1(X) - 1$, and similarly $\beta_3(X') = \beta_3(X) - 1$ by duality, so $\beta_2(X') = \beta_2(X)$. On the other hand, if α is of finite order, β_1 is unchanged by surgery, by duality β_3 is unchanged, so the change in the Euler characteristic must come from an increase in β_2 by 2. \square

Proof of Theorem 6. Form the connected sum $X \# T^4$. This has increased the second Betti number by six. Next, perform surgery on four curves to identify the generators of $\pi_1(T^4)$ with the elements w_i . This does not change the second Betti number because the curves being surgered are of infinite order in $H_1(X \# T^4)$. Since the generators of $\pi_1(T^4)$ commute, the effect of this is that now the four elements w_i commute. Thus the manifold that results from the surgeries has the stated properties. \square

The main algebraic result concerning 4-reduction, and the key to our geometric constructions via Theorem 6, is the following.

Theorem 8. *For $m > 2$ and $n > 2$, the free product $\mathbf{Z}^m * \mathbf{Z}^n$ can be 4-reduced to \mathbf{Z}^{m+n} using $\frac{mn}{6}$ 4-tuples if mn is divisible by 6.*

Proof. If the free product $\mathbf{Z}^m * \mathbf{Z}^n$ can be 4-reduced to \mathbf{Z}^{m+n} using $\frac{mn}{6}$ 4-tuples we will say that the pair (m, n) is realizable. Let \mathcal{R} denote the set of realizable pairs with $m > 2, n > 2$.

First we show that $(3, 4)$, $(3, 6)$, and $(5, 6)$ are in \mathcal{R} .

- Consider first the pair $(3, 4)$. Denote the generators of \mathbf{Z}^3 by $\{x_1, x_2, x_3\}$ and let \mathbf{Z}^4 be generated by $\{y_1, y_2, y_3, y_4\}$. The two 4-tuples $[x_1, y_1, x_2y_2, x_3y_3]$ and $[x_2, x_1y_3, x_3y_2, y_4]$ carry out the desired 4-reduction. For the convenience of the reader we provide the details next, but in subsequent examples similar calculations will be omitted.

We must show that the subgroup U generated by these commutator 4-relations contain all 12 commutators $[x_i, y_j]$. It is helpful to recall that the set of elements in a group which commute with a fixed element forms a subgroup.

From the first 4-relation, $[x_1, y_1, x_2y_2, x_3y_3]$, we see, using the commutators $[x_1, y_1]$, $[x_1, x_2y_2]$, and $[x_1, x_3y_3]$, that the commutators $[x_1, y_1]$, $[x_1, y_2]$, and $[x_1, y_3]$ are in U . The commutators $[y_1, x_2y_2]$ and $[y_1, x_3y_3]$ give that the commutators $[x_2, y_1]$ and $[x_3, y_1]$ are in U . The last commutator, $[x_2y_2, x_3y_3]$, we return to momentarily.

From the second 4-relation, $[x_2, x_1y_3, x_3y_2, y_4]$, we have first the commutator $[x_2, y_3] \in U$. Then, from the previous relation $([x_2y_2, x_3y_3])$ it follows that $[x_3, y_2] \in U$. Next, that the commutators $[x_2, y_2]$ and $[x_2, y_4]$ are in U follow immediately. From the commutator $[x_1y_3, x_3y_2]$ we see that $[x_3, y_3] \in U$ (since we already had that $[x_1, y_2]$ is in U). From $[x_1y_3, y_4]$ we have $[x_1, y_4] \in U$. The commutator $[x_3y_2, y_4]$ gives the last needed commutator, $[x_3, y_4]$.

- In the case of the pair $(3, 6)$, using similar notation, the following three 4-tuples $[x_1, y_1, x_2y_2, x_3y_3]$, $[x_2, x_1y_3, x_3y_4, x_2y_5]$, $[y_6, x_1y_2, x_2x_3, x_3y_4]$ reduce $\mathbf{Z}^3 * \mathbf{Z}^6$.
- Finally, for $(5, 6)$, using similar notation, the following five 4-tuples suffice:
 $[x_1, y_1, x_2y_2, x_3y_3]$, $[x_2, y_3, x_3y_4, x_4y_5]$, $[x_5, y_6, x_4y_3, x_1y_2y_5]$, $[x_3, y_5, x_5y_6, x_1y_4]$,
 $[x_1y_1, x_2y_4, x_4y_6, x_5y_2]$.

For the general case of (m, n) , assume first that m is divisible by 6. Using the realization of $(3, 4)$ we can realize $(6, 4)$ and have already realized $(6, 3)$. (Separate the six generators into two groups of three and make each set commute with the other four using the construction used for $(3, 4)$.) Combining these, we can realize $(6k, 3)$ and $(6k, 4)$ for any k . Now, combining these we can realize $(6k, 3a + 4b)$ for any a and b . But all integers greater than 2, other than 5, can be written as $3a + 4b$ for some a and b .

In the case that neither m nor n are divisible by 6, we can assume 3 divides m and we want to realize $(3k, n)$. Notice that n must be even. Since we can realize $(3, 4)$ and $(3, 6)$, we can realize $(3k, 4)$ and $(3k, 6)$ for all k . Thus we can realize $(3k, 4a + 6b)$ for all a and b , but $4a + 6b$ realizes all even integers greater than 3. \square

5. BASIC REALIZING EXAMPLES

We begin with the exceptional cases of $n = 3$ and $n = 5$ and then move on to a set of basic examples for which $h(n) = C(n, 2) + \epsilon_n$. In the next section we note that these basic examples can be used to construct the necessary examples for the proof of Theorem 1.

- $n = 3$: Start with the 4-torus, T^4 , with $\beta_2(T^4) = 6$. Surgery on a single curve representing a generator of $\pi_1(T^4)$ results in a manifold M with $\pi_1(M) = \mathbf{Z}^3$ and $H_2(M) = \mathbf{Z}^3$.
- $n = 5$: Begin with $X = F_2 \times F_4$ with $\beta_2(X) = 10$ and π_1 generated by $\{x_1, x_2\}$ and $\{y_1, y_2, y_3, y_4\}$. Perform a surgery to identify y_3 and y_4 , so that the group is generated by $\{x_1, x_2, y_1, y_2, y_3\}$. Notice that y_1 and y_2 commute, as follows from the original surface commutator relationship $[y_1, y_2][y_3, y_4] = 1$. This surgery, since it is along an element of infinite order in H_1 , does not change $H_2(X)$. Hence, it only remains to arrange that the pairs of elements $\{y_1, y_3\}$ and $\{y_2, y_3\}$ commute. Performing surgery on a (rationally) null homologous curve raises β_2 by two, so performing surgeries to kill these two commutators raises the rank of $H_2(X)$ by 4, and the resulting 4-manifold M has $H_2(M) = \mathbf{Z}^{14}$ as desired.

We will say that the integer n is *realizable* if there is a closed oriented 4-manifold M_n with $\pi_1(M_n) = \mathbf{Z}^n$ and $\beta_2(M_n) = C(n, 2) + \epsilon_n$. Let \mathcal{S} be the set of realizable integers. We now show that $0, 1, 2, 4, 6, 7, 8, 9, 11, 12 \subset \mathcal{S}$ by describing the construction of realizing 4-manifolds M_n for each of these n .

- $n = 0$: $M_0 = S^4$.
- $n = 1$: $M_1 = S^1 \times S^3$.
- $n = 2$: $M_2 = F_2 \times S^2$. Notice that $C(2, 2) = 1$, so that $\beta_2(M_2) = 2 = C(2, 2) + \epsilon_2$.
- $n = 4$: $M_4 = T^4$.
- $n = 6$: Build M_6 as follows. Let $X = F_2 \times F_4$ with π_1 generated by the 6 elements $\{x_1, x_2\}, \{y_1, y_2, y_3, y_4\}$. Note that $\beta_2(X) = 10$. Apply Theorem 6

to perform the 4-reduction $[y_1, y_2, y_3, y_4]$ and arrive at the 4-manifold M_6 with $\pi_1(M_6) = \mathbf{Z}^6$ and $\beta_2(M_6) = 16 = C(6, 2) + \epsilon_6$ as desired.

- $n = 7$: Begin with $X = F_2 \times F_4 \# T^4$ so $\beta_2(X) = 16$. Let the generators of π_1 be $\{x_1, x_2\}$, $\{y_1, y_2, y_3, y_4\}$, and $\{z_1, z_2, z_3, z_4\}$. Perform surgeries giving $y_1 = z_2$, $y_2 = z_3$, $y_4 = z_4$. Now use Theorem 6 to perform the 4-tuple reduction $[z_1, x_1y_2, x_2y_1, y_3]$. (In checking that this abelianizes the group, use the fact the $[y_1, y_2] = 1$ if and only if $[y_3, y_4] = 1$.) The resulting 4-manifold M_7 has $\pi_1(M_7) = \mathbf{Z}^7$ and $\beta_2(M_7) = 22 = C(7, 2) + \epsilon_7$.
- $n = 8$: Take $X = F_4 \times F_4 \# F_2 \times F_4$, with $\beta_2(X) = 28$ and π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, $\{z_1, z_2\}$ and $\{w_1, w_2, w_3, w_4\}$, respectively. Now perform surgeries to introduce the following relations:

$$\begin{aligned} z_1 &= x_1y_1, \\ z_2 &= x_2y_2, \\ w_1 &= x_3y_2, \\ w_2 &= x_4, \\ w_3 &= x_1y_3, \\ w_4 &= x_2y_4. \end{aligned}$$

Since $[w_1, w_2][w_3, w_4] = 1$ we have that $[x_3y_2, x_4][x_1y_3, x_2y_4] = 1$. Since the x_i commute with the y_i , this implies that $[x_3, x_4][x_1, x_2][y_3, y_4] = 1$. From the surface relation for the x_i $[x_1, x_2][x_3, x_4] = 1$, it then follows that $[y_3, y_4] = 1$. From this and the surface relation for the y_i it follows that $[y_1, y_2] = 1$. Now, the fact that $[z_1, z_2] = 1$ gives that $[x_1, x_2] = 1$, which implies that $[x_3, x_4] = 1$ too. The remaining needed four commutator relations between the x_i and the four commutator relations between the y_i gives a total of 8 needed commutator relations. These are gotten by first considering the relations $[z_1, w_i]$ and then the relations $[z_2, w_i]$. The resulting M_8 has $\pi_1(M_8) = \mathbf{Z}^8$ and $\beta_2(M_8) = 28 = C(8, 2) + \epsilon_8$.

- $n = 9$: Start with $X = T^4 \# T^4 \# T^4$, with $\beta_2(X) = 18$ and π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, and $\{z_1, z_2, z_3, z_4\}$. Perform three surgeries to give the identifications: $z_2 = x_3y_3$, $z_3 = x_4$, and $z_4 = y_4$. Notice that since the 4-tuple relation $[z_1, z_2, z_3, z_4]$ held in the original group, we now have the 4-tuple relation $[z_1, x_3y_3, x_4, y_4]$.

Use Theorem 6 to add the following three more 4-tuple relations, raising β_2 to 36:

$$\begin{aligned} &[z_1, y_1, x_3, x_2y_2], \\ &[z_1x_3, x_1, y_2, x_2y_3], \\ &[x_2y_4, x_4y_3, x_2y_1, x_1y_2y_4]. \end{aligned}$$

The resulting manifold M_9 has $\pi_1(M_9) = \mathbf{Z}^9$ and $\beta_2(M_9) = 36 = C(9, 2) + \epsilon_9$.

- $n = 11$: Start with $X = F_4 \times F_6 \# T^4$, with $\beta_2(X) = 32$ and π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4, y_5, y_6\}$, and $\{z_1, z_2, z_3, z_4\}$. Perform surgery to get the following identifications:

$$\begin{aligned} y_4 &= z_2, \\ y_5 &= z_3, \\ y_6 &= z_4. \end{aligned}$$

This leaves a generating set with eleven elements:

$$\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, y_5, y_6, z_1\}.$$

(Notice that $[x_1, x_2][x_3, x_4] = 1$ and $[y_1, y_2][y_3, y_4] = 1$, since y_5 and y_6 now commute.) Apply Theorem 6 four times to perform the following 4-reductions:

$$\begin{aligned} & [x_1, x_2, x_3, z_1], \\ & [y_1, y_2, y_3, z_1], \\ & [x_4 y_1, y_5, y_3 y_6, y_2 z_1], \\ & [x_1 y_4, x_4 y_6, x_2 y_2 z_1, y_1 y_3]. \end{aligned}$$

The resulting manifold M_{11} has $\pi_1(M_{11}) = \mathbf{Z}^{11}$ and $\beta_2(M_{11}) = 32 + 24 = 56 = C(11, 2) + \epsilon_{11}$.

- $n = 12$: Start with $X = F_4 \times F_4 \# T^4$ with $\beta_2(X) = 24$ and π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, and $\{z_1, z_2, z_3, z_4\}$ as before. Now apply Theorem 6 to add seven 4-tuple relations:

$$\begin{aligned} & [x_1, x_2, x_3, z_1], \\ & [y_1, y_2, y_3, z_1], \\ & [x_1, x_4, z_2, y_1 z_3], \\ & [y_1, y_4, z_3, x_3 z_4], \\ & [x_2 y_2, x_4, y_4 z_1, z_1 z_4], \\ & [x_2 z_3, z_4, y_3, x_3 z_2], \\ & [x_3 z_3, y_4 z_2, x_1 y_2, z_4 x_2 z_2]. \end{aligned}$$

The resulting M_{12} has $\beta_2(M_{12}) = 24 + 42 = 66 = C(12, 2) + \epsilon_{12}$ and $\pi_1(M_{12}) = \mathbf{Z}^{12}$ as desired.

6. CONSTRUCTING MORE EXAMPLES: THE PROOF OF THEOREM 1

Theorem 9. *If $m \in \mathcal{S}$, $n \in \mathcal{S}$, $(m, n) \in \mathcal{R}$, and if one of m , $m - 1$, n , or $n - 1$ is congruent to 0 modulo 4, then $m + n \in \mathcal{S}$.*

Proof. The stated mod 4 condition together with the fact that $mn \equiv 0 \pmod{6}$ assures that $C(n + m, 2) + \epsilon_{n+m} = (C(n, 2) + \epsilon_n) + (C(m, 2) + \epsilon_m) + mn$. Thus, one can build the desired M_{m+n} by performing mn surgeries on $M_m \# M_n \# \frac{mn}{6} T^4$; that is, by performing $\frac{mn}{6}$ 4-reductions as in Theorem 6. \square

We have that $\{0, 1, 2, 4, 6, 7, 8, 9, 11, 12\} \subset \mathcal{S}$. Furthermore, all pairs (n, m) with $mn \equiv 0 \pmod{6}$ and $m \geq 3$ and $n \geq 3$ are in \mathcal{R} .

Using Theorem 9 and the pair $(4, 6) \in \mathcal{R}$ gives $10 \in \mathcal{S}$. Similarly, using the pair $(4, 9) \in \mathcal{R}$ gives $13 \in \mathcal{S}$. The pair $(6, 8) \in \mathcal{R}$ shows that $14 \in \mathcal{S}$. The pair $(6, 9) \in \mathcal{R}$ shows that $15 \in \mathcal{S}$. The pair $(4, 12) \in \mathcal{R}$ shows that $16 \in \mathcal{S}$. The pair $(8, 9) \in \mathcal{R}$ shows that $17 \in \mathcal{S}$.

Next the pairs $(12, n)$ for $n = 6, \dots, 17$ show that $\{18, \dots, 29\} \subset \mathcal{S}$. Then the pairs $(12, n)$ for $n = 18, \dots, 29$ show that $\{30, \dots, 41\} \subset \mathcal{S}$. Repeating inductively in this way shows that $n \in \mathcal{S}$ for all $n \geq 6$, as desired.

7. FINITELY GENERATED ABELIAN GROUPS

We next use the manifolds constructed in the previous sections as building blocks to prove Theorem 2. Let

$$G = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k \oplus \mathbf{Z}^n$$

where $d_i | d_{i+1}$. Fix a prime p that divides d_1 . Then

$$\text{rk}_{\mathbf{Z}/p} H_1(G; \mathbf{Z}/p) = k + n \text{ and } \text{rk}_{\mathbf{Z}/p} H_2(G; \mathbf{Z}/p) = C(k + n, 2) + k.$$

If X is a closed, oriented 4-manifold with $\pi_1(X) \cong G$, we have $H_1(X; \mathbf{Z}/p) \cong H_1(G; \mathbf{Z}/p)$ and $H_2(X; \mathbf{Z}/p)$ surjects to $H_2(G; \mathbf{Z}/p)$. This gives the lower bound on $q(G)$

$$q(G) \geq 2 - 2(k+n) + C(k+n, 2) + k = 1 - n + C(n+k-1, 2). \quad (7.1)$$

To construct upper bounds, consider the following constructions of closed 4-manifolds X with $\pi_1(X) \cong G$. Let $L(d)$ denote a 3-dimensional lens space with $\pi_1(L(d)) = \mathbf{Z}/d$, and let B_n denote a closed 4-manifold with $\pi_1(B_n) = \mathbf{Z}^n$ and $\chi(B_n) = q(\mathbf{Z}^n)$.

- Let X be the manifold obtained by starting with B_{k+n} and doing surgeries on the first k generators of $\pi_1(B_{k+n}) = \mathbf{Z}^{k+n}$ in such a way as to kill d_1 times the first generator, d_2 times the second, and so forth. Then $\pi_1(X) = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k \oplus \mathbf{Z}^n$ and $\text{rk}_{\mathbf{Q}}(H_2(X; \mathbf{Q})) = \text{rk}_{\mathbf{Q}}(H_2(B_{n+k}; \mathbf{Q}))$. Therefore

$$\chi(X) = 2 - 2n + \text{rk}_{\mathbf{Q}}(H_2(B_{n+k}; \mathbf{Q})).$$

Thus, if $n+k \neq 3, 5$, simplifying yields

$$\chi(X) = 1 - n + C(n+k-1, 2) + k + \epsilon_{n+k}$$

and hence

$$0 \leq q(G) - (1 - n + C(n+k-1, 2)) \leq k + \epsilon_{n+k}. \quad (7.2)$$

- Suppose that $n \geq 1$. Start with

$$Y = ((L(d_1) \# L(d_2) \# \cdots \# L(d_k)) \times S^1) \# B_{k+n-1}.$$

Then perform k surgeries which identify the k generators of the connected sum of lens spaces with the first k generators of $\pi_1(B_{k+n-1})$. These surgeries do not change the rank of $H_2(Y; \mathbf{Q})$. Finally perform $n-1$ surgeries along circles representing the commutator of the S^1 factor and the last $n-1$ generators of $\pi_1(B_{k+n-1})$. Each of these surgeries increase the rank of the second rational homology by 2 since the commutators are nullhomologous. This produces a 4-manifold X with $\pi_1(X) \cong G$ and $\text{rk}_{\mathbf{Q}} H_2(X; \mathbf{Q}) = 2(n-1) + \text{rk}_{\mathbf{Q}} H_2(B_{k+n-1}; \mathbf{Q})$. Hence

$$\chi(X) = \text{rk}_{\mathbf{Q}} H_2(B_{k+n-1}; \mathbf{Q}).$$

Thus, if $n+k-1 \neq 3, 5$, $\chi(X) = C(n+k-1, 2) + \epsilon_{n+k-1}$.

Referring to Equation (7.1), this gives the upper bound

$$0 \leq q(G) - (1 - n + C(n+k-1, 2)) \leq n-1 + \epsilon_{n+k-1} \quad (7.3)$$

(for n, k satisfying $n+k-1 \neq 3, 5$ and $n \geq 1$).

- Consider now the case $n = 0$. Start with the 4-manifold obtained from $((L(d_1) \# \cdots \# L(d_{k-1})) \times S^1) \# B_{k-1}$ by performing surgery to identifying the generators of $\pi_1(B_{k-1})$ with the generators for the lens spaces. This yields a closed 4-manifold Y with $\pi_1(Y) \cong \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_{k-1} \oplus \mathbf{Z}$. Surgering d_k times the last generator gives a closed 4-manifold X with $\pi_1(X) \cong \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k$ and $\text{rk}_{\mathbf{Q}}(H_2(X; \mathbf{Q})) = \text{rk}_{\mathbf{Q}}(H_2(B_{k-1}; \mathbf{Q}))$. Therefore

$$\chi(X) = 2 + \text{rk}_{\mathbf{Q}}(H_2(B_{k-1}; \mathbf{Q})).$$

Thus when $k-1 \neq 3, 5$, $\chi(X) = 2 + C(k-1, 2) + \epsilon_{k-1}$, and so for $G = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k$,

$$0 \leq q(G) - (1 + C(k-1, 2)) \leq 1 + \epsilon_{k-1}. \quad (7.4)$$

- In two cases the rational homology gives better lower bounds than the \mathbf{Z}/p homology. For $G = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k$, $\mathrm{rk}_{\mathbf{Q}}(H_1(G; \mathbf{Q})) = 0$ and $\mathrm{rk}_{\mathbf{Q}}(H_2(G; \mathbf{Q})) = 0$ and hence $q(G) \geq 2$. Combined with Equation (7.4) this shows that $q(\mathbf{Z}/d) = 2$ and $q(\mathbf{Z}/d_1 \oplus \mathbf{Z}/d_2) = 2$.

The estimates (7.1), (7.3), (7.2), and (7.4) and the discussion of the previous paragraph combine to give a proof of Theorem 2.

8. REMARKS AND QUESTIONS

A variant of $q(G)$ is obtained by defining $p(G)$ to be the smallest value of $\chi(X) - \sigma(X)$ for all closed oriented 4-manifolds X with $\pi_1(X) = G$. Here $\sigma(X)$ denotes the signature of X . Notice that all the examples we constructed for G abelian have signature zero. We conjecture that $p(\mathbf{Z}^n) = 2 - 2n + C(n, 2)$. This guess is motivated by a slight amount of redundancy which occurs in the constructions given above when $\epsilon_n = 1$. For example, in the case of $n = 6$, the construction we gave starts with $F_2 \times F_4$ and uses one 4-reduction to abelianize the fundamental group of F_4 . In particular, the surface relation $[y_1, y_2][y_3, y_4] = 1$ shows that one of the 6 relations coming from the 4-reduction is unnecessary. This extra bit of flexibility may perhaps be used to twist the geometric construction slightly to introduce some signature.

An interesting question is whether the invariant q depends on the category of manifold or choice of geometric structure. For example, one might consider the infimum of $\chi(X)$ over smooth 4-manifolds or topological 4-manifolds or even 4-dimensional Poincaré complexes with $\pi_1(X) = G$. The manifolds we constructed for $G = \mathbf{Z}^n$ are smooth and the lower bounds are homotopy invariants, so that for $G = \mathbf{Z}^n$ the value of q is independent of the category.

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